Chapter 2 Calculus of functions of several variables

Sec 2.1 Real-valued and vector-valued functions of several variables

- **Def** A function $f: U \underset{\text{"subset of"}}{\subseteq} \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a:

  real-valued function if $n = 1$

  vector-valued function if $n \geq 2$

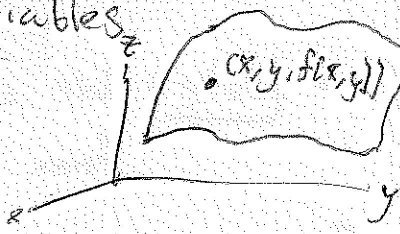
Examples

- distance function $d(x,y,z) = \sqrt{x^2+y^2+z^2}$ is a real-valued function that measures the distance from (x,y,z) to the origin.
- vector equation of a line, $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ is vector-valued.

- For a vector-valued function $F: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ we write $F(x_1,\dots,x_m) = (F_1(x_1,\dots,x_m), \dots, F_n(x_1,\dots,x_m))$ where the $F_i: \mathbb{R}^m \longrightarrow \mathbb{R}$ are real-valued functions called the component functions of F.

  To simplify notation, we let $\vec{x} = (x_1,\dots,x_m)$, so $F(\vec{x}) = (F_1(\vec{x}), \dots, F_n(\vec{x}))$.

Sec 2.2 Graphs of Functions of several variables



- The graph of $y = f(x)$ is a curve in $\mathbb{R}^2$
- The graph of $z = f(x,y)$ is a surface in $\mathbb{R}^3$

- **Def** The level set of value c of a function $f: U \subset \mathbb{R}^m \longrightarrow \mathbb{R}$ is the set of all points $(x_1,\dots,x_m)$ where $f(x_1,\dots,x_m) = c$
- If $m = 2$, level curve = level set
- If $m = 3$, level surface = level set

- **Def** For $m = 2$, level curves form a contour diagram

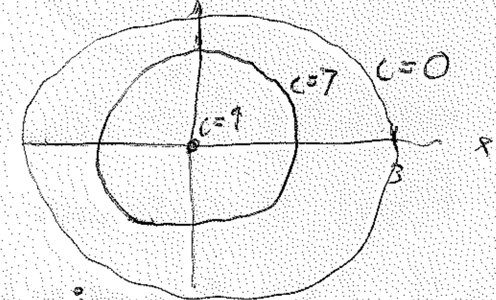
  (think: elevation map)

Example Sketch $z = 9 - x^2 - y^2$

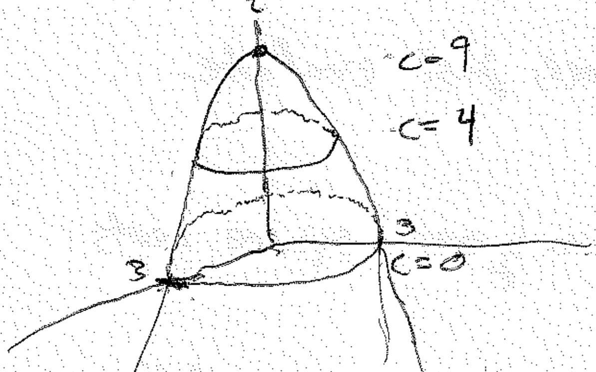
Soln level curves: $9 - x^2 - y^2 = c \longrightarrow x^2 + y^2 = 9 - c$

$c = 9 \rightarrow$ the point $(0,0)$
$c < 9 \rightarrow$ circle of radius $\sqrt{9-c}$
$c > 9 \rightarrow$ no level curves

Contour diagram:



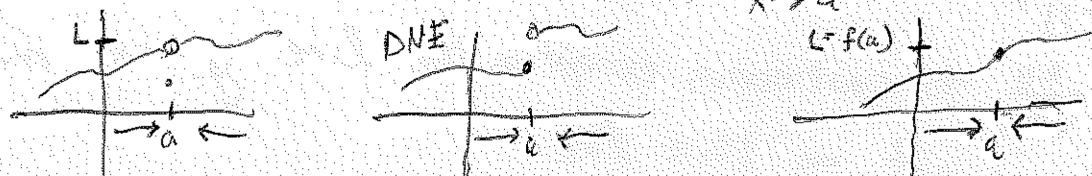
Use to sketch surface:



Sec 2.3 Limits and continuity

- Recall: One variable, we say $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$

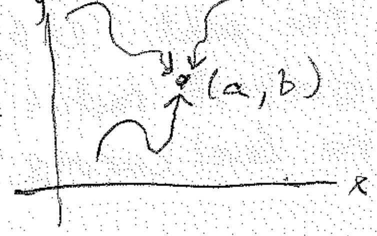
and $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

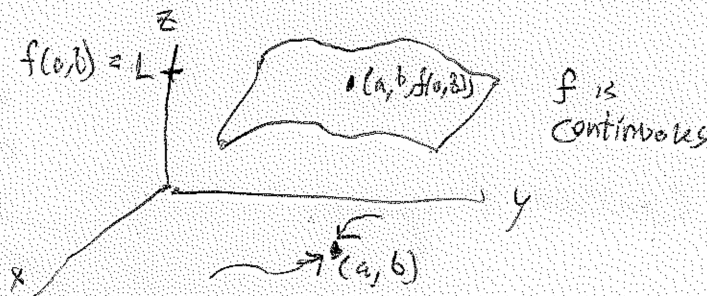


There are only two ways in which we can approach $x = a$.

- Two variables. $\lim_{(x,y) \to (a,b)} f(x,y) = L$ means $f(x,y)$ approaches L as (x,y) gets closer to the point (a,b) (regardless of what $f(a,b)$ is)

- For this limit to exist, $f(x,y)$ must approach the same value regardless of how we approach (a,b).





- Two main tools for evaluating limits/showing they don't exist:
- Def A function $f(x,y)$ is continuous at (a,b) if

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ (so there are no holes/breaks in the graph of f at (a,b))

(1) So if we know a function is continuous, just evaluate $f(a,b)$ to find $\lim_{(x,y)\to(a,b)} f(x,y)$.

- When is $f(x,y)$ continuous? Avoid division by zero, square roots of negative numbers, etc.

(2) If we find two paths along which $f(x,y)$ approaches different values, then the limit doesn't exist.

Examples

(1) $\lim_{(x,y)\to(5,1)} \underbrace{\frac{xy}{x+y}}_{f} = f(5,1) = \frac{5}{6}$.

$f(x,y) = \frac{xy}{x+y}$ is continuous except on the line $y=-x$, and $(5,1)$ is not on this line.

(2) $f(x,y) = \frac{x^2y}{x^4+y^2}$ is not continuous at $(0,0)$, but does $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ exist?

Soln Along the y-axis ($x=0$),

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{y^2} = \lim_{(x,y)\to(0,0)} 0 = 0$$

[In fact, the limit is 0 along any line $y=mx$ through $(0,0)$]

But along the parabola $y=x^2$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{2x^4} = \frac{1}{2} \neq 0,$$

so the limit does not exist.

- To compute the limit of a vector-valued function, just compute the limit of each component function.

Example $\lim_{(x,y,z)\to(1,2,0)} \langle x+y, y^2, xz \rangle = \langle 3, 4, 0 \rangle$.

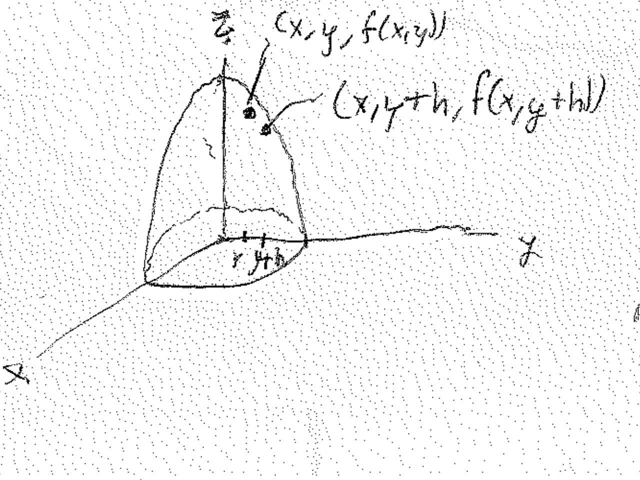
- Continuity: $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$

$f : \mathbb{R}^m \to \mathbb{R}^n$
$\vec{x} = (x_1, \dots, x_m)$
$\vec{a} = (a_1, \dots, a_m)$

Sec 2.4 Derivatives

Recall $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ = rate of change of $f(x)$ as x changes

- How quickly does a surface rise/fall as we move in the positive x or y direction?



$$\frac{\text{change in height}}{\text{change in } y} = \frac{f(x,y+h)-f(x,y)}{h}$$

This leads us to

- **Def** the partial derivatives of $f(x,y)$ with respect to x and y (wrt):

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} =$$ rate of change of $f(x,y)$ as x changes and y is held fixed

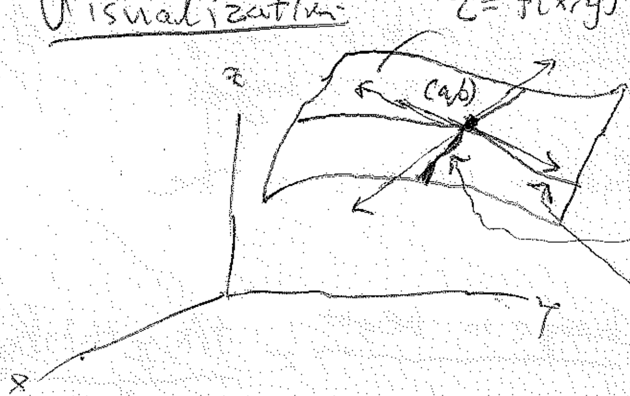
$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} =$$ rate of change of $f(x,y)$ as y changes and x is held fixed.

- Similarly, for $f: U \subseteq \mathbb{R}^m \to \mathbb{R}$, the partial derivative of f wrt x_i is $\frac{\partial f}{\partial x_i}(x_1,\dots,x_m) = f_{x_i}(x_1,\dots,x_m) = \lim_{h\to 0} \frac{f(x_1,\dots,x_i+h,\dots,x_m)-f(x_1,\dots,x_m)}{h}$

- To compute, regard all variables except one as constants and apply single-variable derivative rules.

Example
$$f(x,y,z) = xe^{xy} - \sin(y^2+z^2)$$
$$f_x = e^{xy} + xye^{xy}$$
$$f_y = x^2e^{xy} - 2y\cos(y^2+z^2)$$
$$f_z = -2z\cos(y^2+z^2)$$

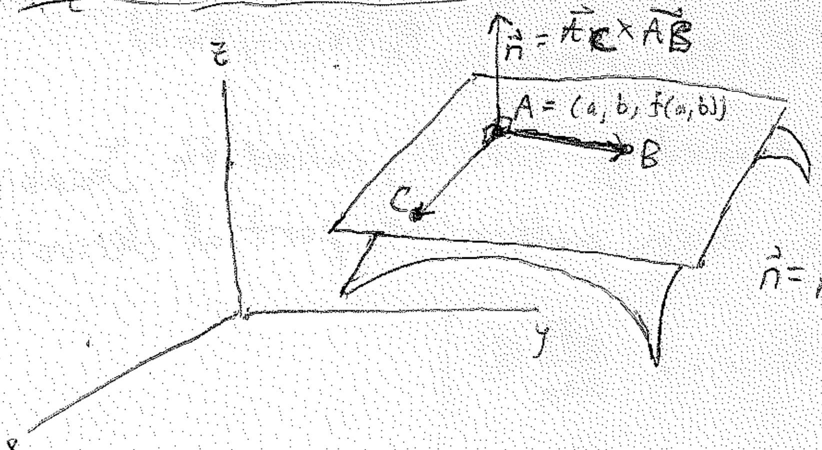
Visualization: $z = f(x,y)$



$f_x(a,b)$ = slope of tangent line to this curve

$f_y(a,b)$ = slope of tangent line to this curve

- <u>Equation of tangent plane to $z=f(x,y)$ at (a,b)</u>:



$$\vec{AB} = \langle 0, 1, f_y(a,b)\rangle$$
$$\vec{AC} = \langle 1, 0, f_x(a,b)\rangle$$
$$\vec{n} = \vec{AC} \times \vec{AB} = \langle -f_x(a,b), -f_y(a,b), 1\rangle$$

Equation of tangent plane is $\vec{n} \cdot \langle x-a, y-b, z-f(a,b)\rangle = 0$.
After simplifying we have:

- Tangent plane to $z=f(x,y)$ at (a,b) is

$$\boxed{z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

- This is also called the <u>linear approximation</u> of f at (a,b).
- $f_x(a,b)(x-a) + f_y(a,b)(y-b) = f_x(a,b)\,dx + f_y(a,b)\,dy$ is the <u>differential</u> of f at (a,b), and gives the error in the approximation.
- <u>Note</u> the similarity to the tangent line equation, $y = f(a) + f'(a)(x-a)$.

<u>Example</u> Tangent plane to $f(x,y) = 1 - x^2 - 2y^2$ at $(1,1)$.

<u>Soln</u> $f_x = -2x \longrightarrow f_x(1,1) = -2$

$f_y = -4y \longrightarrow f_y(1,1) = -4$

$f(1,1) = 1-1-2 = -2$

$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$z = -2 - 2(x-1) - 4(y-1)$, or $z = 4 - 2x - 4y$

- We can use this to approximate values of f near $(1,1)$:

$f(0.96, 1.02) \approx -2 - 2(.96-1) - 4(1.02-1) = -2$

actual value: $f(0.96, 1.02) = -2.0024$

- **Def** (derivative of vector-valued function $F: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$)

Write $F(x_1, \dots, x_m) = (F_1(x_1, \dots, x_m), \dots, F_n(x_1, \dots, x_m))$

The <u>derivative of F</u>, denoted by $DF(\vec{x})$, called the Jacobian matrix of F, is the $n \times m$ matrix of partial derivatives

$$DF(\vec{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\vec{x}) & \frac{\partial F_1}{\partial x_2}(\vec{x}) & \cdots & \frac{\partial F_1}{\partial x_m}(\vec{x}) \\ \frac{\partial F_2}{\partial x_1}(\vec{x}) & \frac{\partial F_2}{\partial x_2}(\vec{x}) & \cdots & \frac{\partial F_2}{\partial x_m}(\vec{x}) \\ \vdots & \vdots & & \\ \frac{\partial F_n}{\partial x_1}(\vec{x}) & \frac{\partial F_n}{\partial x_2}(\vec{x}) & \cdots & \frac{\partial F_n}{\partial x_m}(\vec{x}) \end{bmatrix}$$

- $DF(\vec{a})$ is the best linear approximation of F near $\vec{a}$,

- <u>Special cases</u>:
  - $f: \mathbb{R} \to \mathbb{R}$: $Df(x)$ is our usual $f'(x)$
  - $f: \mathbb{R}^m \to \mathbb{R}$. $Df(\vec{x})$ is the $1 \times m$ matrix that we interpret as a vector in $\mathbb{R}^m$, $Df(\vec{x}) = \left\langle \frac{\partial f}{\partial x_1}(\vec{x}), \dots, \frac{\partial f}{\partial x_m}(\vec{x}) \right\rangle$

    In this case $Df(\vec{x})$ is called the <u>gradient of f</u> and is denoted by $\nabla f(\vec{x})$ or grad $f(\vec{x})$.

<u>Example</u> Compute $DF(\vec{x})$ for $F(x,y,z) = (e^{x+yz}, x^2+1, \sin(y+z), 4y)$

<u>Soln</u> $F: \mathbb{R}^3 \to \mathbb{R}^4$ so $DF(\vec{x})$ is 4×3.

$$DF(\vec{x}) = \begin{bmatrix} e^{x+yz} & ze^{x+yz} & ye^{x+yz} \\ 2x & 0 & 0 \\ 0 & \cos(y+z) & \cos(y+z) \\ 0 & 4 & 0 \end{bmatrix}$$

Sec 2.6 Derivative properties:

- Let $F, G : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be differentiable at $\vec{a} \in U$. Then

$$D(F \pm G)(\vec{a}) = DF(\vec{a}) \pm DG(\vec{a})$$

- For $c \in \mathbb{R}$, $D(cF)(\vec{a}) = c\,DF(\vec{a})$
- For $f, g : U \subseteq \mathbb{R}^m \to \mathbb{R}$ differentiable at $\vec{a} \in U$,

$$\nabla(fg)(\vec{a}) = g(\vec{a})\nabla f(\vec{a}) + f(\vec{a})\nabla g(\vec{a}), \text{ and}$$

- if $g(\vec{a}) \neq 0$, $\nabla\left(\frac{f}{g}\right)(\vec{a}) = \dfrac{g(\vec{a})\nabla f(\vec{a}) - f(\vec{a})\nabla g(\vec{a})}{g(\vec{a})^2}$
- For $\vec{v}, \vec{w} : U \subseteq \mathbb{R} \to \mathbb{R}^n$ differentiable at $a \in U$,

$$(\vec{v} \cdot \vec{w})'(a) = \vec{v}\,'(a) \cdot \vec{w}(a) + \vec{v}(a) \cdot \vec{w}\,'(a)$$

- For $\vec{v}, \vec{w} : U \subseteq \mathbb{R} \to \mathbb{R}^3$ differentiable at $a \in U$,

$$(\vec{v} \times \vec{w})'(a) = \vec{v}\,'(a) \times \vec{w}(a) + \vec{v}(a) \times \vec{w}\,'(a) \quad \text{(order matters!)}$$

- Chain rule: Let $F : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$, $G : V \subseteq \mathbb{R}^n \to \mathbb{R}^p$ + assumptions in text.

Then $D(G \circ F)(\vec{a}) = DG(F(\vec{a})) \cdot DF(\vec{a})$. [notation: $(G \circ F)(\vec{x}) = G(F(\vec{x}))$]

Example For $F(x,y) = (x^3 + y, e^{xy}, 2 + xy)$,
$G(u,v,w) = (u^2 + v, uv + w^3)$, we have

$$D(G \circ F) = DG(F(0,1)) \cdot DF(0,1) = DG(1,1,2) \cdot DF(0,1)$$

$$= \begin{bmatrix} 2u & 1 & 0 \\ v & u & 3w^2 \end{bmatrix}_{(1,1,2)} \begin{bmatrix} 3x^2 & 1 \\ ye^{xy} & xe^{xy} \\ y & x \end{bmatrix}_{(0,1)}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 12 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 13 & 1 \end{bmatrix}.$$

Recall If $y = f(x)$, $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

- Change of variables / chain rule - Special case:

Setup: • f and the x_i's are real-valued functions
• $f = f(x_1, \dots, x_m)$ and each $x_i = x_i(t_1, \dots, t_m)$

Then $$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial f}{\partial x_m}\frac{\partial x_n}{\partial t_i}.$$

Example Let $f(u,v,w) = u^2 + v^3 e^w$ where $u = \sin(x+y+z)$, $v = x^2 e^y$, $w = z$

Then $$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$
$$= 2u\cos(x+y+z) + 3v^2 e^w \cdot x^2 e^y + v^3 e^w \cdot 0 = \dots$$

Example $f(x,y) = xe^{xy}$, $x = t^2$, $y = t^{-1}$

Then $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= (e^{xy} + xye^{xy})(2t) + x^2 e^{xy}(-t^{-2}).$$

Sec 2.5 Paths and curves

- Def A path in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is a function $\vec{c} : [a,b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) ($\pm\infty$ allowed)

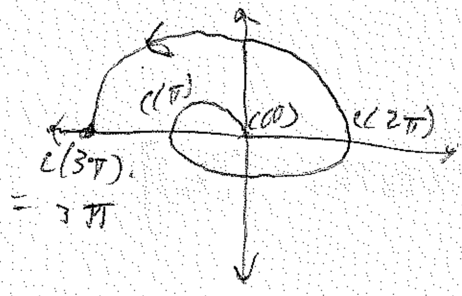
The graph of $\vec{c}$ is a curve in $\mathbb{R}^2$ (or $\mathbb{R}^3$)

$\vec{c}$ is also called a parametrization of the curve.

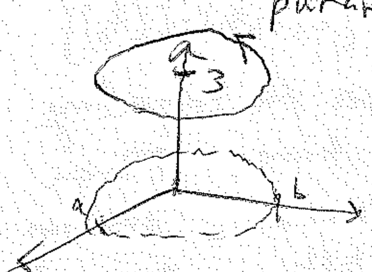
Example From Ch. 1, $\vec{c}(t) = \vec{a} + t\vec{v}$, $t \in (-\infty, \infty)$ is a path, namely the line through $\vec{a}$ in the direction of $\vec{v}$.

Example $\vec{c}(t) = (t\cos t, t\sin t)$, $t \in [0, 3\pi]$

Because $t^2\cos^2 t + t^2\sin^2 t = t^2$, $(t, c(t))$ lies on the circle of radius t. $\vec{c}$ parametrizes this curve:

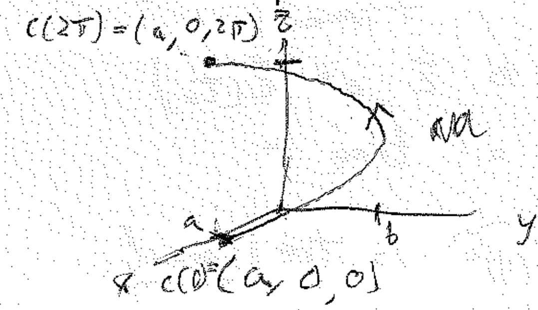


Example $\vec{c}(t) = (a\cos t, b\sin t, 3)$, $t \in [0, 2\pi]$ parametrizes an ellipse



If instead we let $t \in [0, 4\pi]$, the curve is the same, but these are two different paths.

Example $c(t) = (a\cos t, b\sin t, t)$, $t \in [0, 2\pi]$



- Terminology: $\left.\begin{array}{l} c(\vec{a}) = \text{initial point} \\ c(\vec{b}) = \text{terminal point} \end{array}\right] = \underline{\text{end points}}$

The direction corresponding to increasing t-values gives the <u>positive orientation</u>. The opposite direction gives the <u>negative orientation</u>.

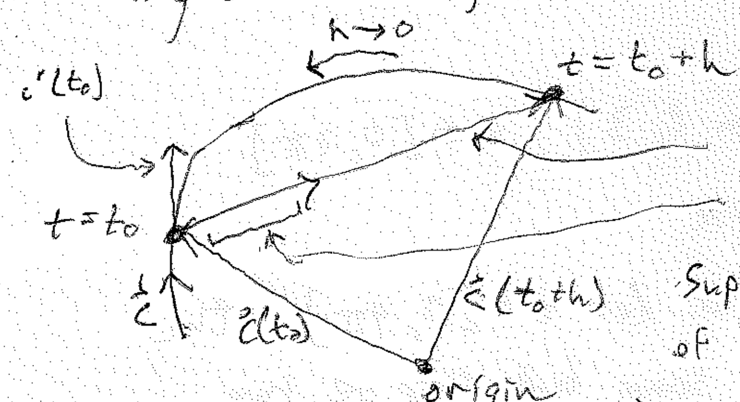
Example $\vec{c}_1(t) = (r\cos t, r\sin t)$, $t \in [0, 2\pi]$
$\vec{c}_2(t) = (r\cos t, -r\sin t)$, $t \in [0, 2\pi]$
are parametrizations of a circle of radius r
positive orientation for c_1 (resp., c_2) = counterclockwise (resp. clockwise)

- **Notation** For $\vec{c}(t) = (x(t), y(t), z(t))$,

$$D\vec{c}(t_0) = \begin{bmatrix} x'(t_0) \\ y'(t_0) \\ z'(t_0) \end{bmatrix} = (x'(t_0), y'(t_0), z'(t_0)) = \vec{c}\,'(t)\Big|_{t=t_0}$$

- **Meaning of $c'(t_0)$:**

By definition, $\vec{c}\,'(t_0) = \lim_{h\to 0} \frac{1}{h}\left[\vec{c}(t_0+h) - \vec{c}(t_0)\right]$



displacement vector $\vec{c}(t_0+h) - \vec{c}(t_0)$

scaled version $\frac{1}{h}\left[\vec{c}(t_0+h) - \vec{c}(t_0)\right]$

Suppose $\vec{c}(t)$ describes the trajectory of a particle over time. Then

$\vec{c}(t_0+h) - \vec{c}(t_0) =$ position at time t_0+h $-$ position at time t_0

so $\dfrac{\vec{c}(t_0+h) - \vec{c}(t_0)}{h} = \dfrac{\text{displacement vector}}{\text{time}}$

Hence,

- **Def** • $c'(t_0)$ is called a tangent vector to $\vec{c}$ at $\vec{c}(t_0)$
  - $\vec{v}(t) = \vec{c}\,'(t)$ is the velocity at time t
  - $\|\vec{v}(t)\| = \|\vec{c}\,'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$ is the speed.
  - $\vec{a}(t) = \vec{v}\,'(t) = \vec{c}\,''(t)$ is the acceleration at time t.

Example $c_1(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$

$c_2(t) = (\cos 2t, \sin 2t)$, $t \in [0, \pi]$

$v_1(t) = c_1'(t) = (-\sin t, \cos t)$, so the speed is $\sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

$v_2(t) = c_2'(t) = (-2\sin 2t, 2\cos 2t)$, so the speed is

$\sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2} = \sqrt{4} = 2.$

Sec 2.7 Gradient and directional derivative

- **Def** The gradient of f, $\nabla f = \text{grad } f$, at $\vec{x} = (x_1, \dots, x_m) \in U$ of $f: U \subseteq \mathbb{R}^m \to \mathbb{R}$ is $\nabla f(\vec{x}) = \langle \frac{\partial f}{\partial x_1}(\vec{x}), \dots, \frac{\partial f}{\partial x_m}(\vec{x}) \rangle$

Example: Let $f(x,y) = x^2 + 3xy$. Then $\nabla f(x,y) = \langle 2x+3y, 3x \rangle$.

- **Recall** f_x and f_y give the rate of change of $f(x,y)$ as we move in the positive x and y directions.

What about the rate of change for any other directions?

- **Def** Let $f: U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be differentiable.
The directional derivative of f at $p = (a,b)$ in the direction of the unit vector $\vec{u} = (u, v)$ is

$$\nabla_{\vec{u}} = D_{\vec{u}} f(a,b) = \frac{d}{dt} f(\underbrace{p + t\vec{u}}_{\text{line through } p \text{ in direction of } \vec{u}.})\Big|_{t=0}$$

- **Theorem:** $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$.
proof: For $\vec{r}(t) = p + t\vec{u}$, apply chain rule to $f(r(t))$.

Example: Compute the directional derivative of $f(x,y) = x^2 + 3xy$ in the direction of $\langle 3,4 \rangle$ at the point $p = (2,-1)$.

Soln: First we need a unit vector: $\vec{u} = \frac{\langle 3,4 \rangle}{\|\langle 3,4 \rangle\|} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

Using the definition, we compute

$$f(p + t\vec{u}) = f((2,-1) + t(\tfrac{3}{5}, \tfrac{4}{5})) = f(2 + \tfrac{3}{5}t, -1 + \tfrac{4}{5}t)$$
$$= (2 + \tfrac{3}{5}t)^2 + 3(2 + \tfrac{3}{5}t)(-1 + \tfrac{4}{5}t)$$
$$= \tfrac{9}{5}t^2 + \tfrac{27}{5}t - 2$$

Thus, $\frac{d}{dt}(f(p+tu)) = \frac{18}{5}t + \frac{27}{5}$.

Plug in $t=0$ to get $D_{\vec{u}} f(2,-1) = \frac{27}{5}$.

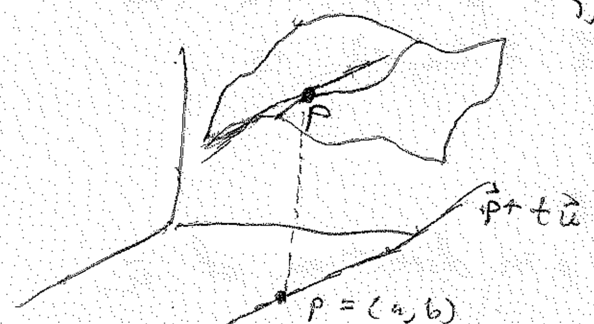
More easily, we could have applied the theorem:

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}.$$

$\nabla f = (2x+3y, 3x)$, so $\nabla f(2,-1) = (1,6)$, so

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = (1,6) \cdot \left(\tfrac{3}{5}, \tfrac{4}{5}\right) = \frac{3}{5} + \frac{24}{5} = \frac{27}{5}.$$

- Meaning: $D_{\vec{u}} f(a,b)$ = rate of change of f per unit distance in the direction of $\vec{u}$.
  = slope of tangent line to $\vec{c}(t)$ at P.



In the previous example, if ~~an~~ ant is at $(2,-1)$ and moves 1 unit in the direction of $\vec{u}$, $f(x,y)$ will increase by about $\frac{27}{5}$.

- Maximum rate of change: Let $f: U \subseteq \mathbb{R}^2$ (or $\mathbb{R}^3$) $\to \mathbb{R}$ be differentiable and assume $\nabla f(\vec{x}) = \vec{0}$ for $\vec{x} \in U$.
- The direction of the largest rate of increase of f at $\vec{x}$ is given by the vector $\nabla f(\vec{x})$, and the largest rate of increase is $\|\nabla f(\vec{x})\|$.
- If $\nabla f(\vec{x}) = \vec{0}$, then $D_{\vec{u}} f(\vec{x}) = 0$ in all directions.
- The direction of the largest rate of decrease is $-\nabla f(\vec{x})$ with magnitude $\|\nabla f(x)\|$.
- The rate of change in directions perpendicular to $\nabla f(\vec{x})$ is 0.

Partial proof (1st and last properties):

$$D_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} = \|\nabla f(\vec{x})\| \, \|\vec{u}\| \cos\theta$$
$$= \|\nabla f(\vec{x})\| \cos\theta \qquad (\|\vec{u}\|=1)$$
$$\leq \|\nabla f(\vec{x})\| \qquad (|\cos\theta| \leq 1)$$

If $\vec{u}$ is perpendicular to $\nabla f(\vec{x})$, ~~then $\theta = \frac{\pi}{2}$~~
then $D_u f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u} = 0$.

Example Let $f(x,y) = e^{-(x^2+y^2)}$. Find the direction of the largest rate of increase of f at $(1,1)$.

Soln This direction is given by

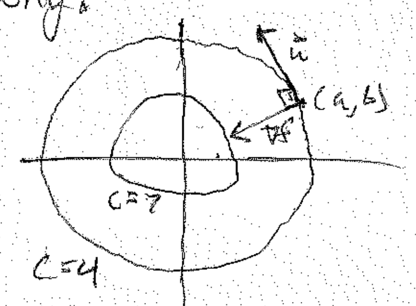
$$\nabla f(1,1) = \left(-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)}\right)\Big|_{(1,1)} = \left(-2e^{-2}, -2e^{-2}\right),$$

The maximum rate of change of f is then

$$\|\nabla f(1,1)\| = \sqrt{(-2e^{-2})^2 + (-2e^{-2})^2} = \sqrt{8e^{-4}}.$$

- The gradient vector is perpendicular to level curves

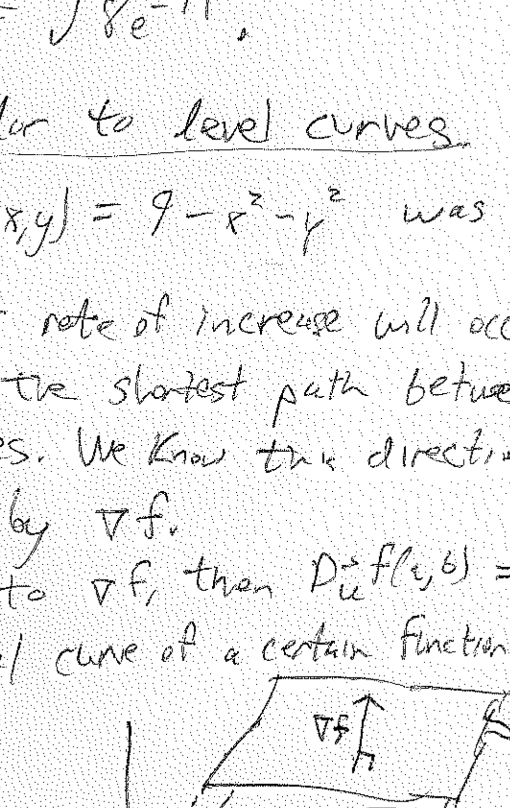
Why? The contour diagram for $f(x,y) = 9 - x^2 - y^2$ was



The largest rate of increase will occur if we take the shortest path between level curves. We know this direction is given by ∇f.

Also note that if $\vec{u}$ is perpendicular to ∇f, then $D_{\vec{u}} f(a,b) = 0$, since we're staying on the same level curve of a certain function value.

- The gradient vector is perpendicular to level surfaces as well.



<u>Example</u> $\sin(xy) - 2\cos(yz) = 0$ is a level surface of value 0 of the function $f(x,y,z) = \sin(xy) - 2\cos(yz)$.

Find the equation of its tangent plane at the point $p = (\frac{\pi}{2}, 1, \frac{\pi}{3})$.

<u>Soln</u> A normal vector is

$$\vec{n} = \nabla f(p) = \left\langle y\cos(xy),\ x\cos(xy) + 2z\sin(yz),\ 2y\sin(yz)\right\rangle \Big|_{(\frac{\pi}{2}, 1, \frac{\pi}{3})}$$

$$= \left\langle 0, \frac{\pi\sqrt{3}}{3}, \sqrt{3} \right\rangle$$

The tangent plane equation is $\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$:

$$\left(0, \frac{\pi\sqrt{3}}{3}, \sqrt{3}\right) \cdot \left(x - \frac{\pi}{2}, y - 1, z - \frac{\pi}{3}\right) = 0$$